Extremal Bounds for Bootstrap Percolation in the Hypercube

Natasha Morrison[1] Jonathan A. Noel[1,2]

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Abstract

The *r-neighbour bootstrap process* on a graph G starts with an initial set A_0 of "infected" vertices and, at each step of the process, a healthy vertex becomes infected if it has at least r infected neighbours (once a vertex becomes infected, it remains infected forever). If every vertex of G eventually becomes infected, then we say that A_0 *percolates*.

We prove a conjecture of Balogh and Bollobás which says that, for fixed r and $d \to \infty$, every percolating set in the d-dimensional hypercube has cardinality at least $\frac{1+o(1)}{r}\binom{d}{r-1}$. We also prove an analogous result for multidimensional rectangular grids. Our proofs exploit a connection between bootstrap percolation and a related process, known as *weak saturation*. In addition, we improve the best known upper bound for the minimum size of a percolating set in the hypercube. In particular, when $r = 3$, we prove that the minimum cardinality of a percolating set in the d-dimensional hypercube is $\left\lceil \frac{d(d+3)}{6} \right\rceil + 1$ for all $d \geq 3$.

1 Introduction

Given a positive integer r and a graph G, the *r-neighbour bootstrap process* begins with an initial set of "infected" vertices of G and, at each step of the process, a vertex becomes infected if it has at least r infected neighbours. More formally, if A_0 is the initial set of infected vertices, then the set of vertices that are infected after the jth step of the process for $j \geq 1$ is defined by

$$A_j := A_{j-1} \cup \{v \in V(G) : |N_G(v) \cap A_{j-1}| \geq r\},$$

where $N_G(v)$ denotes the neighbourhood of v in G. We say that A_0 *percolates* if $\bigcup_{j=0}^{\infty} A_j = V(G)$. Bootstrap percolation was introduced by Chalupa, Leath and Reich [14] as a mathematical simplification of existing dynamic models of ferromagnetism, but it has also found

[1]Mathematical Institute, University of Oxford, UK. E-mail: {morrison,noel}@maths.ox.ac.uk.
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applications in the study of other physical phenomena such as crack formation and hydrogen mixtures (see Adler and Lev [1]). In addition, advances in bootstrap percolation have been highly influential in the study of more complex processes including, for example, the Glauber dynamics of the Ising model [20].

The main extremal problem in bootstrap percolation is to determine the minimum cardinality of a set which percolates under the r-neighbour bootstrap process on G; we denote this by $m(G, r)$. An important case is when G is the *d-dimensional hypercube* Q_d; ie., the graph with vertex set $\{0, 1\}^d$ in which two vertices are adjacent if they differ in exactly one coordinate. Balogh and Bollobás [4] (see also [8, 9]) made the following conjecture.

Conjecture 1.1 (Balogh and Bollobás [4]). *For fixed $r \geq 3$ and $d \to \infty$,*

$$m(Q_d, r) = \frac{1 + o(1)}{r}\binom{d}{r-1}.$$

The upper bound of Conjecture 1.1 is not difficult to prove. Simply let A_0 consist of all vertices on "level $r-2$" of Q_d and an approximate Steiner system on level r, whose existence is guaranteed by an important theorem of Rödl [25]; see Balogh, Bollobás and Morris [8] for more details. Note that, under certain conditions on d and r, the approximate Steiner system in this construction can be replaced with an exact Steiner system (using, for example, the celebrated result of Keevash [19]). In this special case, the percolating set has cardinality $\frac{1}{r}\binom{d}{r-1} + \binom{d}{r-2}$ which yields

$$m(Q_d, r) \leq \frac{d^{r-1}}{r!} + \frac{d^{r-2}(r+2)}{2r(r-2)!} + O\left(d^{r-3}\right). \tag{1.2}$$

Lower bounds have been far more elusive; previously, the best known lower bound on $m(Q_d, r)$ for fixed $r \geq 3$ was only linear in d (see Balogh, Bollobás and Morris [8]). In this paper, we prove Conjecture 1.1.

Theorem 1.3. *For $d \geq r \geq 1$,*

$$m(Q_d, r) \geq 2^{r-1} + \sum_{j=1}^{r-1}\binom{d-j-1}{r-j}\frac{j2^{j-1}}{r}$$

where, by convention, $\binom{a}{b} = 0$ when $a < b$.

For fixed $r \geq 3$, Theorem 1.3 implies

$$m(Q_d, r) \geq \frac{d^{r-1}}{r!} + \frac{d^{r-2}(6-r)}{2r(r-2)!} + \Omega\left(d^{r-3}\right),$$

which differs from the upper bound in (1.2) by an additive term of order $\Theta\left(d^{r-2}\right)$. We will also provide a recursive upper bound on $m(Q_d, r)$, which improves on the second order term of (1.2). For $r = 3$, we combine this recursive bound with some additional arguments to show that Theorem 1.3 is tight in this case.

Theorem 1.4. *For $d \geq 3$, we have $m(Q_d, 3) = \left\lceil \frac{d(d+3)}{6} \right\rceil + 1$.*

In order to prove Theorem 1.3, we will exploit a relationship between bootstrap percolation and the notion of weak saturation introduced by Bollobás [10]. Given fixed graphs G and H, we say that a spanning subgraph F of G is *weakly* (G, H)*-saturated* if the edges of $E(G) \setminus E(F)$ can be added to F, one edge at a time, in such a way that each edge completes a copy of H when it is added. The main extremal problem in weak saturation is to determine the *weak saturation number* of H in G defined by

$$\operatorname{wsat}(G, H) := \min \{|E(F)| \colon F \text{ is weakly } (G, H)\text{-saturated}\}.$$

Weak saturation is very well studied (see, e.g. [3, 17, 18, 21, 22, 24]). Our proof of Theorem 1.3 relies on the following bound, which is easy to prove:

$$m(G, r) \geq \frac{\operatorname{wsat}(G, S_{r+1})}{r} \tag{1.5}$$

where S_{r+1} denotes the star with $r+1$ leaves. A slightly stronger version of (1.5) is stated and proved in the next section. We obtain an exact expression for the weak saturation number of S_{r+1} in the hypercube.

Theorem 1.6. *If $d \geq r \geq 0$, then*

$$\operatorname{wsat}(Q_d, S_{r+1}) = r2^{r-1} + \sum_{j=1}^{r-1} \binom{d-j-1}{r-j} j2^{j-1}.$$

Note that Theorem 1.3 follows directly from this theorem and (1.5). More generally, we determine the weak saturation number of S_{r+1} in the d-dimensional $a_1 \times \cdots \times a_d$ grid, denoted by $\prod_{i=1}^d [a_i]$. We state this result here in the case $d \geq r$; an even more general result is expressed later in terms of a recurrence relation.

Theorem 1.7. *For $d \geq r \geq 1$ and $a_1, \ldots, a_d \geq 2$,*

$$\operatorname{wsat}\left(\prod_{i=1}^d [a_i], S_{r+1}\right) = \sum_{\substack{S \subseteq [d] \\ |S| \leq r-1}} \left(\prod_{i \in S} (a_i - 2)\right) \left((r - |S|)2^{r-|S|-1} + \sum_{j=1}^{r-|S|-1} \binom{d-|S|-j-1}{r-|S|-j} j2^{j-1}\right).$$

Observe that a lower bound on $m\left(\prod_{i=1}^d [a_i], r\right)$ follows from Theorem 1.7 and (1.5). To our knowledge, the combination of Theorem 1.7 and (1.5) implies all of the known lower bounds on the cardinality of percolating sets in multidimensional grids. In particular, it implies the (tight) lower bounds

$$m\left([n]^d, d\right) \geq n^{d-1},$$

and

$$m\left(\prod_{i=1}^{d}[a_i],2\right) \geq \left\lceil \frac{\sum_{i=1}^{d}(a_i-1)}{2} \right\rceil + 1. \tag{1.8}$$

established in [23] and [4], respectively.

An important motivation for Conjecture 1.1 stems from its potential applications in a probabilistic setting. The most well studied problem in bootstrap percolation is to estimate the *critical probability* at which a randomly generated set of vertices in a graph G becomes likely to percolate. To be more precise, for $p \in [0,1]$, suppose that A_0^p is a subset of $V(G)$ obtained by including each vertex randomly with probability p independently of all other vertices and define

$$p_c(G,r) := \inf\left\{p : \mathbb{P}\left(A_0^p \text{ percolates}\right) \geq 1/2\right\}.$$

The problem of estimating $p_c\left([n]^d, r\right)$ for fixed d and r and $n \to \infty$ was first considered by Aizenman and Lebowitz [2] and subsequently studied in [6, 12, 13, 15, 16]. This rewarding line of research culminated in a paper of Balogh, Bollobás, Duminil-Copin and Morris [5] in which $p_c\left([n]^d, r\right)$ is determined asymptotically for all fixed values of d and $2 \leq r \leq d$.

Comparably, far less is known about the critical probability when d tends to infinity. In this regime, the main results are due to Balogh, Bollobás and Morris in the case $r = d$ [7] and $r = 2$ [8]. In the latter paper, the extremal bound (1.8) was applied to obtain precise asymptotics for $p_c\left([n]^d, 2\right)$ whenever $d \gg \log(n) \geq 1$. In contrast, very little is known about the critical probability for fixed $r \geq 3$ and $d \to \infty$. For example, the logarithm of $p_c(Q_d, 3)$ is not even known to within a constant factor (see [8]). As was mentioned in [9], a stumbling block in obtaining good estimates for $p_c(Q_d, r)$ when $d \to \infty$ has been the lack of an asymptotically tight lower bound $m(Q_d, r)$. In this paper, we provide such a bound.

The rest of the paper is organised as follows. In the next section, we outline our approach to proving Theorems 1.3 and 1.7 and establish some preliminary lemmas. In Section 3, we prove Theorem 1.6. We then determine $\mathrm{wsat}\left(\prod_{i=1}^{d}[a_i], S_{r+1}\right)$ in full generality in Section 4 using similar ideas (which become somewhat more cumbersome in the general setting). In Section 5, we provide constructions of small percolating sets in the hypercube and prove Theorem 1.4. Finally, we conclude the paper in Section 6 by stating some open problems related to our work.

2 Preliminaries

We open this section by proving the following lemma, which improves on (1.5) for graphs with vertices of degree less than r (including, for example, the graph $\prod_{i=1}^{d}[a_i]$ for $d < r$).

Lemma 2.1. *Let G be a graph and let F be a spanning subgraph of G such that the set*

$$A_0 := \{v \in V(G) : d_F(v) \geq \min\{r, d_G(v)\}\}$$

percolates with respect to the r-neighbour bootstrap process on G. Then F is weakly (G, S_{r+1})-saturated.

Proof. By hypothesis, we can label the vertices of G by $v_1, \ldots, v_n$ in such a way that

- $\{v_1, \ldots, v_{|A_0|}\} = A_0$, and
- for $|A_0|+1 \leq i \leq n$, the vertex v_i has at least r neighbours in $\{v_1, \ldots, v_{i-1}\}$.

Let us show that F is weakly (G, S_{r+1})-saturated. We begin by adding to F every edge of $E(G) \setminus E(F)$ which is incident to a vertex in A_0 (one edge at a time in an arbitrary order). For every vertex $v \in A_0$, we have that either

- there are at least r edges of F incident to v, or
- every edge of G incident with v is already present in F.

Therefore, every edge of $E(G) \setminus E(F)$ incident to a vertex in A_0 completes a copy of S_{r+1} when it is added.

Now, for each $i = |A_0|+1, \ldots, n$ in turn, we add every edge incident to v_i which has not already been added (one edge at a time in an arbitrary order). Since v_i has at least r neighbours in $\{v_1, \ldots, v_{i-1}\}$ and every edge incident to a vertex in $\{v_1, \ldots, v_{i-1}\}$ is already present, we get that every such edge completes a copy of S_{r+1} when it is added. The result follows. $\square$

For completeness, we will now deduce (1.5) from Lemma 2.1.

Proof of (1.5). Let A_0 be a set of cardinality $m(G, r)$ which percolates with respect to the r-neighbour bootstrap process on G and let F be a spanning subgraph of G such that $d_F(v) \geq \min\{d_G(v), r\}$ for each $v \in A_0$. Note that this can be achieved by adding at most r edges per vertex of A_0 and so we can assume that $|E(F)| \leq r|A_0| = rm(G, r)$. By Lemma 2.1, F is weakly (G, S_{r+1})-saturated and so

$$\operatorname{wsat}(G, S_{r+1}) \leq |E(F)| \leq rm(G, r)$$

as required. $\square$

We turn our attention to determining the weak saturation number of stars in hypercubes and, more generally, in multidimensional rectangular grids. To prove an upper bound on a weak saturation number, one only needs to construct a *single* example of a weakly saturated graph of small size. Our main tool for proving the lower bound is the following linear algebraic lemma of Balogh, Bollobás, Morris and Riordan [9]. A major advantage of this lemma is that it allows us to prove the lower bound in a constructive manner as well. We include a proof for completeness.

Lemma 2.2 (Balogh, Bollobás, Morris and Riordan [9])**.** *Let G and H be graphs and let W be a vector space. Suppose that $\{f_e : e \in E(G)\}$ is a collection of vectors in W such that for every copy H' of H in G there exists non-zero coefficients $\{c_e : e \in E(H')\}$ such that $\sum_{e\in E(H')} c_e f_e = 0$. Then*

$$\mathrm{wsat}(G, H) \geq \dim(\mathrm{span}\,\{f_e : e \in E(G)\}).$$

Proof. Let F be a weakly (G, H)-saturated graph and define $m := |E(G) \setminus E(F)|$. By definition of F, we can label the edges of $E(G) \setminus E(F)$ by $e_1, \ldots, e_m$ in such a way that, for $1 \leq i \leq m$, there is a copy H_i of H in $F_i := F \cup \{e_1, \ldots, e_i\}$ containing the edge e_i. By the hypothesis, we get that

$$f_{e_i} \in \mathrm{span}\,\{f_e : e \in E(H_i) \setminus \{e_i\}\} \subseteq \mathrm{span}\,\{f_e : e \in E(F_i) \setminus \{e_i\}\}$$

for all i. Therefore,

$$|E(F)| \geq \dim\left(\mathrm{span}\,\{f_e : e \in E(F)\}\right) = \dim\left(\mathrm{span}\,\{f_e : e \in E\left(F_1\right)\}\right)$$

$$= \cdots = \dim\left(\mathrm{span}\,\{f_e : e \in E\left(F_m\right)\}\right) = \dim\left(\mathrm{span}\,\{f_e : e \in E(G)\}\right).$$

The result follows. □

Lemma 2.2 was proved in a more general form and applied to a percolation problem in multidimensional square grids in [9]. It was also used by Morrison, Noel and Scott [21] to determine $\mathrm{wsat}\left(Q_d, Q_m\right)$ for all $d \geq m \geq 1$. We remark that the general idea of applying the notions of dependence and independence in weak saturation problems is also present in the works of Alon [3] and Kalai [18], where techniques involving exterior algebra and matroid theory were used to prove a tight lower bound on $\mathrm{wsat}(K_n, K_k)$ conjectured by Bollobás [11]. For a more recent application of exterior algebra and matroid theory to weak saturation problems, see the paper of Pikhurko [24].

3 The Hypercube Case

Our goal in this section is to prove Theorem 1.6. This will settle the case $a_1 = \cdots = a_d = 2$ of Theorem 1.7 and, as discussed earlier, imply Theorem 1.3 via (1.5). First, we require some definitions.

Definition 3.1. Given $k \geq 1$, an index $i \in [k]$ and $x \in \mathbb{R}^k$, let x_i denote the ith coordinate of x. The *support* of x is defined by $\mathrm{supp}(x) := \{i \in [k] : x_i \neq 0\}$.

Definition 3.2. The *direction* of an edge $e = uv \in E\left(Q_d\right)$ is the unique index $i \in [d]$ such that $u_i \neq v_i$. Given a vertex $v \in V(Q_d)$, we define $e(v, i)$ to be the unique edge in direction i that is incident to v.

Note that each edge of Q_d receives two labels (one for each of its endpoints). Our approach will make use of the following simple linear algebraic fact.

Lemma 3.3. *Let $k \geq \ell \geq 0$ be fixed. Then there exists a subspace X of $\mathbb{R}^k$ of dimension $k - \ell$ such that $|\mathrm{supp}(x)| \geq \ell + 1$ for every $x \in X \setminus \{0\}$.*

Proof. Define X to be the span of a set $\{v_1, \ldots, v_{k-\ell}\}$ of unit vectors of $\mathbb{R}^k$ chosen independently and uniformly at random with respect to the standard Lebesgue measure on the unit sphere S^{k-1}. Given a fixed subspace W of $\mathbb{R}^k$ of dimension at most ℓ and $1 \leq i \leq k-\ell$, the space

$$\mathrm{span}\,(W \cup \{v_1, \ldots, v_{i-1}\})$$

has dimension less than k. Thus, the unit sphere of this space has measure zero in S^{k-1} and so, with probability one, $v_i \notin \mathrm{span}\,(W \cup \{v_1, \ldots, v_{i-1}\})$. It follows that $\dim(X) = k - \ell$ and $X \cap W = \{0\}$ almost surely. In particular, if we let $T \subseteq [k]$ be a fixed set of cardinality ℓ and define

$$W_T := \left\{x \in \mathbb{R}^k : \mathrm{supp}(x) \subseteq T\right\},$$

then $X \cap W_T = \{0\}$ almost surely. Since there are only finitely many sets $T \subseteq [k]$ of cardinality ℓ, we can assume that X is chosen so that $X \cap W_T = \{0\}$ for every such set. This completes the proof. $\square$

In the appendix, we provide an explicit (ie. non-probabilistic) example of a vector space X satisfying Lemma 3.3. The following lemma highlights an important property of the space X.

Lemma 3.4. *Let $k \geq \ell \geq 0$ and let X be a subspace of $\mathbb{R}^k$ of dimension $k - \ell$ such that $|\mathrm{supp}(x)| \geq \ell + 1$ for every $x \in X \setminus \{0\}$. For every set $T \subseteq [k]$ of cardinality $\ell + 1$, there exists $x \in X$ with $\mathrm{supp}(x) = T$.*

Proof. Let $T \subseteq [k]$ with $|T| = \ell + 1$. Clearly, the space $\{x \in \mathbb{R}^k : \mathrm{supp}(x) \subseteq T\}$ has dimension $\ell + 1$. Therefore, since $\dim(X) = k - \ell$, there must be a non-zero vector $x \in X$ with $\mathrm{supp}(x) \subseteq T$. However, this inclusion must be equality since $|\mathrm{supp}(x)| \geq \ell + 1$. $\square$

We are now in position to prove Theorem 1.6. For notational convenience, we write

$$w := r2^{r-1} + \sum_{j=1}^{r-1} \binom{d-j-1}{r-j} j2^{j-1}.$$

Also, using Lemma 3.3, let X be a subspace of $\mathbb{R}^d$ of dimension $d - r$ such that $|\mathrm{supp}(x)| \geq r + 1$ for every $x \in X \setminus \{0\}$. We deduce Theorem 1.6 from the following lemma, after which we will prove the lemma itself.

Lemma 3.5. *There is a spanning subgraph F of Q_d and a collection $\{f_e : e \in E(Q_d)\} \subseteq \mathbb{R}^w$ such that*

(Q1) F is weakly (Q_d, S_{r+1})-saturated and $|E(F)| = w$,

(Q2) $\sum_{i=1}^{d} x_i f_{e(v,i)} = 0$ for every $v \in V(Q_d)$ and $x \in X$, and

(Q3) $\operatorname{span}\{f_e : e \in E(Q_d)\} = \mathbb{R}^w$.

Proof of Theorem 1.6. Clearly, the existence of a graph F satisfying (Q1) implies the upper bound $\operatorname{wsat}(Q_d, S_{r+1}) \leq w$. We show that the lower bound follows from (Q2), (Q3) and Lemma 2.2. Note that the edge sets of copies of S_{r+1} in Q_d are precisely the sets of the form $\{e(v,i) : i \in T\}$ where v is a fixed vertex of Q_d and T is a subset of $[d]$ of cardinality $r+1$. By Lemma 3.4 we know that there exists some $x \in X$ with $\operatorname{supp}(x) = T$. By (Q2) we have

$$\sum_{i=1}^{d} x_i f_{e(v,i)} = \sum_{i \in T} x_i f_{e(v,i)} = 0.$$

Therefore, by Lemma 2.2,

$$\operatorname{wsat}(Q_d, S_{r+1}) \geq \dim(\operatorname{span}\{f_e : e \in E(Q_d)\})$$

which equals w by (Q3). The result follows. $\square$

Proof of Lemma 3.5. We proceed by induction on d. We begin by settling some easy boundary cases before explaining the inductive step.

Case 1: $r = 0$.

In this case, $S_{r+1} \simeq K_2$. Also, $w = 0$ and $X = \mathbb{R}^d$. We let F be a spanning subgraph of Q_d with no edges and set $f_e := 0$ for every $e \in Q_d$. It is trivial to check that (Q1), (Q2) and (Q3) are satisfied.

Case 2: $d = r \geq 1$.

In this case, $w = d2^{d-1} = |E(Q_d)|$ and $X = \{0\}$. We define $F := Q_d$ and let $\{f_e : e \in E(Q_d)\}$ be a basis for $\mathbb{R}^w$. Clearly (Q1), (Q2) and (Q3) are satisfied.

Case 3: $d > r \geq 1$.

We begin by constructing F in such a way that (Q1) is satisfied. For $i \in \{0,1\}$, let Q^i_{d-1} denote the subgraph of Q_d induced by $\{0,1\}^{d-1} \times \{i\}$. Note that both Q^0_{d-1} and Q^1_{d-1} are isomorphic to Q_{d-1}. Let F be a spanning subgraph of Q_d such that

- the subgraph F_0 of F induced by $V\left(Q^0_{d-1}\right)$ is a weakly (Q_{d-1}, S_{r+1})-saturated graph of minimum size,
- the subgraph F_1 of F induced by $V\left(Q^1_{d-1}\right)$ is a weakly (Q_{d-1}, S_r)-saturated graph of minimum size, and
- F contains no edge in direction d.

Figure 1 contains a specific instance of this construction. Define $w_0 := \text{wsat}\,(Q_{d-1}, S_{r+1})$ and $w_1 := \text{wsat}\,(Q_{d-1}, S_r)$. By construction, we have $|E(F)| = w_0 + w_1$ which is equal to w by the inductive hypothesis. Let us verify that F is weakly (Q_d, S_{r+1})-saturated. To see this, we add the edges of $E\,(Q_d) \setminus E(F)$ to F in three stages. By construction, we can begin by adding all edges of Q^0_{d-1} which are not present in F_0 in such a way that each edge completes a copy of S_{r+1} in Q^0_{d-1} when it is added. In the second stage, we add all edges of Q_d in direction d one by one in any order. Since every vertex of Q_d has degree $d \geq r+1$ and every edge of Q^0_{d-1} has already been added, we get that every edge added in this stage completes a copy of S_{r+1} in Q_d. Finally, we add the edges of Q^1_{d-1} which are not present in F_1 in such a way that each added edge completes a copy of S_r in Q^1_{d-1}. Since the edges in direction d have already been added, we see that every such edge completes a copy of S_{r+1} in Q_d. Therefore, (Q1) holds.

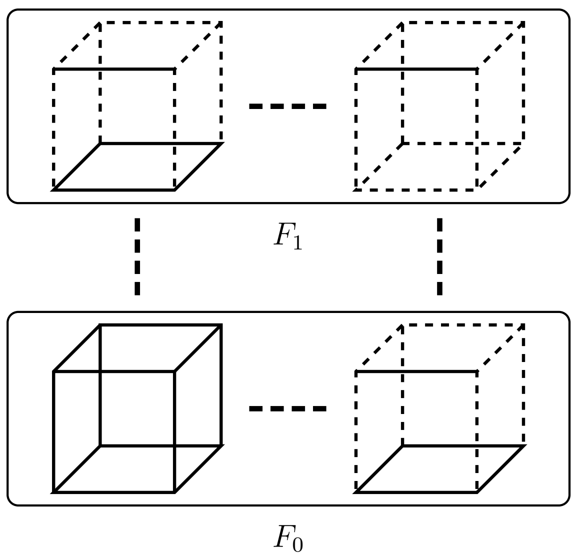

Figure 1: A weakly (Q_5, S_4)-saturated graph F constructed inductively from a weakly (Q_4, S_4)-saturated graph F_0 and a weakly (Q_4, S_3)-saturated graph F_1, each of which is also constructed inductively.

Thus, all that remains is to construct $\{f_e : e \in E\,(Q_d)\}$ in such a way that (Q2) and (Q3) are satisfied. Let $\pi : X \to \mathbb{R}^{d-1}$ be the standard projection defined by $\pi : (x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1})$. Let $z \in X$ be an arbitrary vector such that $d \in \text{supp}\,(z)$ (such a vector exists by Lemma 3.4) and let $T_z : X \to X$ be the linear map defined by

$$T_z(x) := x - \frac{x_d}{z_d} z$$

for $x \in X$. Define

$$X_0 := \pi\,(T_z(X)) \text{ and}$$
$$X_1 := \pi(X).$$

Clearly, $\ker(T_z) = \operatorname{span}\{z\}$ and, since every $x \in X \setminus \{0\}$ has $|\operatorname{supp}(x)| \geq r+1 \geq 2$, we have $\ker(\pi) = \{0\}$. This implies that X_0 has dimension $d-r-1$ and that X_1 has dimension $d-r$. Also, by construction, we have that $|\operatorname{supp}(x)| \geq r+1$ for every non-zero $x \in X_0$ and $|\operatorname{supp}(x)| \geq r$ for every non-zero $x \in X_1$.

Therefore, by the inductive hypothesis, there exists $\left\{f_e^0 : e \in E\left(Q_{d-1}^0\right)\right\}$ in $\mathbb{R}^{w_0}$ and $\left\{f_e^1 : e \in E\left(Q_{d-1}^1\right)\right\}$ in $\mathbb{R}^{w_1}$ such that

(Q2.0) $\sum_{i=1}^{d-1} x_i f^0_{e(v,i)} = 0$ for every $v \in V\left(Q_{d-1}^0\right)$ and $x \in X_0$,

(Q2.1) $\sum_{i=1}^{d-1} x_i f^1_{e(v,i)} = 0$ for every $v \in V\left(Q_{d-1}^1\right)$ and $x \in X_1$,

(Q3.0) $\operatorname{span}\left\{f_e^0 : e \in E\left(Q_{d-1}^0\right)\right\} = \mathbb{R}^{w_0}$, and

(Q3.1) $\operatorname{span}\left\{f_e^1 : e \in E\left(Q_{d-1}^1\right)\right\} = \mathbb{R}^{w_1}$.

We will define the vectors $\{f_e : e \in E(Q_d)\} \subseteq \mathbb{R}^{w_0} \oplus \mathbb{R}^{w_1} \simeq \mathbb{R}^w$ satisfying (Q2) and (Q3) in three stages. First, if $e \in E\left(Q_{d-1}^0\right)$, then we set

$$f_e := f_e^0 \oplus 0.$$

Next, for each edge of the form $e = e(v,d)$ for $v \in V\left(Q_{d-1}^0\right)$, we let

$$f_e := -\frac{1}{z_d} \sum_{i=1}^{d-1} z_i f_{e(v,i)} \tag{3.6}$$

(recall the definition of z above). Finally, if $e = uv \in E\left(Q_{d-1}^1\right)$, then we let $e' = u'v'$ where u' and v' are the unique neighbours of u and v in $V\left(Q_{d-1}^0\right)$ and define

$$f_e := f_{e'}^0 \oplus f_e^1.$$

It is easily observed that $\dim\left(\operatorname{span}\{f_e : e \in E(Q_d)\}\right) = w_0 + w_1 = w$ by (Q3.0), (Q3.1) and the construction of f_e given above. Therefore, (Q3) holds.

Finally, we prove that (Q2) is satisfied. First, let $v \in V\left(Q_{d-1}^0\right)$ and let $x \in X$ be arbitrary. Define $x^\dagger := T_z(x)$ and note that $d \notin \operatorname{supp}\left(x^\dagger\right)$. We have

$$\sum_{i=1}^{d} x_i f_{e(v,i)} = \sum_{i=1}^{d-1} x_i^\dagger f_{e(v,i)} + \frac{x_d}{z_d} \sum_{i=1}^{d} z_i f_{e(v,i)}$$

by definition of T_z. Both of the sums on the right side are zero by (Q2.0) and (3.6). Now, suppose that $v \in V\left(Q_{d-1}^1\right)$ and let v' be the unique neighbour of v in $V\left(Q_{d-1}^0\right)$. Given $x \in X$, we have

$$\sum_{i=1}^{d} x_i f_{e(v,i)} = \sum_{i=1}^{d} x_i f_{e(v',i)} + \sum_{i=1}^{d-1} x_i \left(0 \oplus f^1_{e(v,i)}\right)$$

which is zero by (Q2.1) and the fact that $\sum_{i=1}^{d} x_i f_{e(v',i)} = 0$, which was proven above (as $v' \in V\left(Q_{d-1}^0\right)$). Therefore, (Q2) holds. This completes the proof of the lemma. □

4 General Grids

Our objective in this section is to determine the weak saturation number of S_{r+1} in $\prod_{i=1}^{d}[a_i]$ in full generality. We express this weak saturation number in terms of the following recurrence relation.

Definition 4.1. Let d and r be integers such that $0 \le r \le 2d$ and let $a_1, \dots, a_d \ge 2$. Define $w_r(a_1, \dots, a_d)$ to be

- $0,$ if $r = 0$;
- $\displaystyle\sum_{j=1}^{d}(a_j - 1)\prod_{i \ne j} a_i,$ if $r = 2d$;
- $d2^{d-1},$ if $a_1 = \dots = a_d = 2$ and $d + 1 \le r \le 2d - 1$;
- $\displaystyle r2^{r-1} + \sum_{j=1}^{r-2}\binom{d-j-1}{r-j} j2^{j-1},$ if $a_1 = \dots = a_d = 2$ and $1 \le r \le d$; and
- $w_r(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots a_d) + w_{r-1}(a_1, \dots, a_{i-1}, a_{i+1}, \dots a_d)$ $\displaystyle + \sum_{\substack{S \subseteq [d]\setminus\{i\} \\ |S| \ge 2d-r}} 2^{|S|}\prod_{j \notin S}(a_j - 2),$ if $1 \le r \le 2d - 1$ and $a_i \ge 3$.

We prove the following.

Theorem 4.2. *For $0 \le r \le 2d$ and $a_1, \dots, a_d \ge 2$, we have*

$$\mathrm{wsat}\left(\prod_{i=1}^{d}[a_i], S_{r+1}\right) = w_r(a_1, \dots, a_d).$$

Before presenting the proof let us remark that, for $d \ge r$, the expression in Theorem 1.7 satisfies the recurrence in Definition 4.1. Therefore, Theorem 4.2 implies Theorem 1.7. Let $a_1, \dots, a_d \ge 2$ and define $G := \prod_{i=1}^{d}[a_i]$. In proving of Theorem 4.2, we employ an inductive approach similar to the one used in the proof of Theorem 1.6. The main difference is that a vertex v of G may be incident to either one or two edges in direction $i \in [d]$ depending on whether or not $v_i \in \{1, a_i\}$. With this in mind, we define a labelling of the edges of G.

Definition 4.3. Say that an edge $e = uv \in E(G)$ in direction $i \in [d]$ is *odd* if $\min\{u_i, v_i\}$ is odd and *even* otherwise. We label e by $e(v, 2i - 1)$ if e is odd and $e(v, 2i)$ if e is even.

Note that each edge of G receives two labels, one for each of its endpoints.

Definition 4.4. For $v \in V(G)$, define $I_v^G := \{j \in [2d] : e(v, j) \in E(G)\}$.

We are now in position to prove Theorem 4.2. Using Lemma 3.3, we let X be a subspace of $\mathbb{R}^{2d}$ of dimension $2d-r$ such that $|\text{supp}(x)| \geq r+1$ for every $x \in X \setminus \{0\}$. Define $w := w_r(a_1, \ldots, a_d)$. As with the proof of Theorem 1.6, we state a lemma from which we deduce Theorem 4.2, and then we prove the lemma.

Lemma 4.5. *There is a spanning subgraph F of G and a collection $\{f_e : e \in E(G)\} \subseteq \mathbb{R}^w$ such that*

(G1) F *is weakly* (G, S_{r+1})*-saturated and* $|E(F)| = w$*,*

(G2) $\sum_{i=1}^{2d} x_i f_{e(v,i)} = 0$ *for every* $v \in V(G)$ *and* $x \in X$ *such that* $\text{supp}(x) \subseteq I_v^G$*, and*

(G3) $\text{span}\{f_e : e \in E(G)\} = \mathbb{R}^w$*.*

Proof of Theorem 4.2. First observe that the existence of a graph F satisfying (G1) implies $\text{wsat}(G, S_{r+1}) \leq w$. To obtain a matching lower bound, we apply Lemma 2.2 as we did in the hypercube case. The edge sets of copies of S_{r+1} in G are the sets of the form $\{e(v,i) : i \in T\}$, where $v \in V(G)$ and T is a subset of I_v^G of cardinality $r+1$. By applying Lemma 3.4 together with (G2), we see that the conditions of Lemma 2.2 are satisfied. Thus by (G3), $\text{wsat}(G, S_{r+1}) \geq w$. □

Proof of Lemma 4.5. We proceed by induction on $|V(G)|$. We begin with the boundary cases.

Case 1: $r = 0$.

In this case, $S_{r+1} \simeq K_2$. Also, $w = 0$ and $X = \mathbb{R}^{2d}$. We let F be a spanning subgraph of G with no edges and set $f_e := 0$ for every $e \in Q_d$. Properties (G1), (G2) and (G3) are satisfied trivially.

Case 2: $r = 2d \geq 2$.

In this case, $w = |E(G)|$ and $X = \{0\}$. We define $F := G$ and let $\{f_e : e \in E(G)\}$ be a basis for $\mathbb{R}^w$. Clearly (G1), (G2) and (G3) are satisfied.

Case 3: $a_1 = \ldots = a_d = 2$ and $1 \leq r \leq 2d-1$.

In this case, G is isomorphic to Q_d and every edge of G is odd. First, suppose that $d+1 \leq r \leq 2d-1$. Then we have $w = |E(G)|$ and we define $F := G$ and let $\{f_e : e \in E(G)\}$ be a basis for $\mathbb{R}^w$.

On the other hand, if $1 \leq r \leq d$, then we let X' be the subspace of X consisting of all vectors x of X such that every element of $\text{supp}(x)$ is odd. It is not hard to show that X' has dimension $d-r$ and that every vector $x \in X'$ has $|\text{supp}(x)| \geq r+1$. Thus, we are done by Lemma 3.5.

Case 4: $a_i \geq 3$ for some $i \in [d]$ and $1 \leq r \leq 2d-1$.

Without loss of generality, assume that $a_d \geq 3$. Define

$$G_1 := \prod_{i=1}^{d-1} [a_i] \times [a_d - 1], \text{ and}$$

$$G_2 := G \setminus G_1.$$

Observe that every vertex of G_2 has a unique neighbour in $V(G_1)$. The edges with one endpoint in G_1 and the other in G_2 will play a particular role in the proof. We define

$$\tau := \begin{cases} 2d-1 & a_d - 1 \text{ is odd} \\ 2d & a_d - 1 \text{ is even,} \end{cases}$$

and we write $\bar{\tau}$ for the unique element of $\{2d-1, 2d\} \setminus \{\tau\}$. Observe that for $v \in V(G_2)$, we have that $\bar{\tau} \notin I_v^G$, and that $I_v^{G_2} = I_v^G \setminus \{\tau\}$. On the other hand, if $v \in V(G_1)$, then

$$I_v^{G_1} = \begin{cases} I_v^G \setminus \{\tau\} & \text{if } v_d = a_d - 1, \\ I_v^G & \text{otherwise.} \end{cases}$$

Define

$$Y := \{v \in V(G_1) : v_d = a_d - 1 \text{ and } d_{G_1}(v) < r\}.$$

It is not hard to see that

$$|Y| = \sum_{\substack{S \subseteq [d-1] \\ |S| \geq 2d-r}} 2^{|S|} \prod_{j \notin S} (a_j - 2).$$

For brevity we write $y := |Y|$ and

$$w_1 := \mathrm{wsat}(G_1, S_{r+1}),$$

$$w_2 := \mathrm{wsat}(G_2, S_r).$$

We construct a graph F satisfying (G1). Define F to be a spanning subgraph of G such that

- the subgraph F_1 of F induced by $V(G_1)$ is a weakly (G_1, S_{r+1})-saturated graph of minimum size,
- the subgraph F_2 of F induced by $V(G_2)$ is a weakly (G_2, S_r)-saturated graph of minimum size, and
- an edge e from $V(G_1)$ to $V(G_2)$ is contained in F if and only if e is of the form $e(v, \tau)$ for $v \in Y$.

Applying the inductive hypothesis and Definition 4.1, we see that $|E(F)|= w_1 + w_2 + y = w$, as required. To see that F is weakly (G, S_{r+1})-saturated, we add the edges of $E(G)\backslash E(F)$ to F in three stages. First, by definition of F_1, we can add the edges that are not present in $E(F_1)$ in such a way that every added edge completes a copy of S_{r+1} in G_1. Next, we can add the edges of the form $e(v, \tau)$, where $v \notin Y$ and $v_d = a_d - 1$, in any order. By definition of Y, we see that every such v has at least r neighbours in G_1. As every edge in $E(G_1)$ has already been added, the addition of $e(v, \tau)$ completes a copy of S_{r+1} in G. Finally, we add the edges of G_2 that are not present in F_2 in such a way that each added edge completes a copy of S_r in G_2. Every such edge completes a copy of S_{r+1} in G since every vertex in G_2 has a neighbour in G_1 and every edge between G_1 and G_2 is already present. Thus, (G1) holds.

It remains to find a collection $\{f_e : e \in E(G)\}$ satisfying (G2) and (G3). Let $\pi : X \to \mathbb{R}^{2d-2}$ be the projection defined by $\pi : (x_1, \ldots, x_{2d}) \mapsto (x_1, \ldots, x_{2d-2})$. Let z be a fixed vector of X such that $\bar{\tau} \in \operatorname{supp}(z)$ and define $T_z : X \to X$ by

$$T_z(x) := x - \frac{x_{\bar{\tau}}}{z_{\bar{\tau}}}.$$

Define $X_1 := X$ and $X_2 := \pi(T_z(X))$. Since $\ker(T_z) = \operatorname{span}\{z\}$ and $\ker(\pi) = \{0\}$ we see that X_2 has dimension $2d - r - 1 = 2(d - 1) - (r - 1)$. Also, by construction, we have $|\operatorname{supp}(x)|\geq r$ for every non-zero $x \in X_2$. By applying the inductive hypothesis to both G_1 and G_2, we can find collections $\{f^1_e : e \in E(G_1)\}$ in $\mathbb{R}^{w_1}$ and $\{f^2_e : e \in E(G_2)\}$ in $\mathbb{R}^{w_2}$ such that

(G2.1) $\sum_{i=1}^{2d} x_i f^1_{e(v,i)} = 0$ for every $v \in V(G_1)$ and $x \in X_1$ with $\operatorname{supp}(x) \subseteq I^{G_1}_v$,

(G2.2) $\sum_{i=1}^{2d-2} x_i f^2_{e(v,i)} = 0$ for every $v \in V(G_2)$ and $x \in X_2$ with $\operatorname{supp}(x) \subseteq I^{G_2}_v$,

(G3.1) $\operatorname{span}\{f^1_e : e \in E(G_1)\} = \mathbb{R}^{w_1}$, and

(G3.2) $\operatorname{span}\{f^2_e : e \in E(G_2)\} = \mathbb{R}^{w_2}$.

Using this, we will now construct a collection $\{f_e : e \in E(G)\} \subseteq \mathbb{R}^{w_1} \oplus \mathbb{R}^{w_2} \oplus \mathbb{R}^y \simeq \mathbb{R}^w$ in four steps. First, for $e \in E(G_1)$, we define

$$f_e := f^1_e \oplus 0 \oplus 0.$$

Let $\{f^3_y : y \in Y\}$ be a basis of $\mathbb{R}^y$. Next, we consider edges $e = uv$, where $v \in V(G_1)$, and $u \in V(G_2)$. If v is in Y, then we let

$$f_e := 0 \oplus 0 \oplus f^3_v.$$

If v is not in Y, then let $z^v \in X$ be a vector such that $\operatorname{supp}(z^v) \subseteq I^G_v$ and $\tau \in \operatorname{supp}(z^v)$, which exists by Lemma 3.3. Define

$$f_e := -\frac{1}{z^v_\tau} \sum_{i\in[2d]\backslash\{\tau\}} z^v_i f_{e(v,i)}. \tag{4.6}$$

Finally if $e = uv \in E(G_2)$, then let $e' = u'v'$ where $u'v'$ are the unique neighbours of u and v in $V(G_1)$ and define

$$f_e := f^1_{e'} \oplus f^2_e \oplus 0.$$

It is clear from (G3.1), (G3.2) and the construction of f_e, that the dimension of $\mathrm{span}\{f_e : e \in E(G)\}$ is $w_1 + w_2 + y = w$. Thus (G3) is satisfied.

It remains to show that (G2) holds. Firstly, suppose $v \in V(G_1)$ and let $x \in X$ be such that $\mathrm{supp}(x) \subseteq I^G_v$. If $v_d < a_d - 1$, then $\sum_{i=1}^{2d} x_i f_{e(v,i)} = 0$ by (G2.1). If $v_d = a_d - 1$ and $v \in Y$, then, by definition of Y, we have $|I^G_v| \le r$ and so it must be the case that $x = 0$ and we are done. Now suppose that $v \notin Y$ and that $v_d = a_d - 1$. Define

$$x^\dagger := x - \frac{x_\tau}{z^v_\tau} z^v.$$

We have,

$$\sum_{i=1}^{2d} x_i f_{e(v,i)} = \sum_{i=1}^{2d} x^\dagger_i f_{e(v,i)} + \frac{x_\tau}{z^v_\tau} \sum_{i=1}^{2d} z^v_i f_{e(v,i)}. \tag{4.7}$$

Note that $\tau \notin \mathrm{supp}(x^\dagger)$ and thus $\mathrm{supp}(x^\dagger) \subseteq I^{G_1}_v$. Therefore, both of the sums on the right side of (4.7) are zero by (G2.1) and (4.6).

Finally, consider $v \in V(G_2)$. Let v' be the unique neighbour of v in $V(G_2)$. Given $x \in X$, with $\mathrm{supp}(x) \subseteq I^G_v$ we have

$$\sum_{i=1}^{2d} x_i f_{e(v,i)} = \sum_{i=1}^{2d} x_i f_{e(v',i)} + \sum_{i=1}^{2d-2} x_i \left(0 \oplus f^2_{e(v,i)} \oplus 0\right).$$

We have that $\sum_{i=1}^{2d} x_i f_{e(v',i)} = 0$ for $v' \in V(G_1)$, as proved above. The second sum on the right side is zero by (G2.2), which is applicable as $\bar{\tau} \notin I^G_v \supseteq \mathrm{supp}(x)$, and so $x \in T_z(X)$. This completes the proof of the lemma. $\square$

5 Upper Bound Constructions

In this section, we prove a recursive upper bound on $m(Q_d, r)$ for general $d \ge r \ge 1$ and then apply it to obtain an exact expression for $m(Q_d, 3)$.

Lemma 5.1. *For $d \ge r \ge 1$,*

$$m\left(Q_d, r\right) \le m\left(Q_{d-r}, r\right) + (r-1)m\left(Q_{d-r}, r-1\right) + \sum_{j=1}^{\lceil r/2 \rceil - 1} \binom{r}{2j+1} m\left(Q_{d-r}, r-2j\right).$$

Proof. Let $d \ge r$ be fixed positive integers. For $1 \le t \le r$, let B_t be a subset of $V\left(Q_{d-r}\right)$ of cardinality $m\left(Q_{d-r}, t\right)$ which percolates with respect to the t-neighbour bootstrap process in Q_{d-r}.

Given $x \in V(Q_d)$, let $[x]_r$ and $[x]_{d-r}$ denote the vectors obtained by restricting x to its first r coordinates and last $d-r$ coordinates, respectively. We partition $\{0,1\}^r$ into $r+1$ sets $L_0, \ldots, L_r$ such that L_i consists of the vectors whose coordinate sum is equal to i. We construct a percolating set A_0 in Q_d. Given $x \in V(Q_d)$, we include x in A_0 if one of the following holds:

- $[x]_r \in L_1$ and either
  - $[x]_r = (1, 0, \ldots, 0)$ and $[x]_{d-r} \in B_r$.
  - $[x]_r \neq (1, 0, \ldots, 0)$ and $[x]_{d-r} \in B_{r-1}$.
- $[x]_r \in L_{2j+1}$ for some $1 \leq j \leq \lceil r/2 \rceil - 1$ and $[x]_{d-r} \in B_{r-2j}$.

It is clear that

$$|A_0| = m\left(Q_{d-r}, r\right) + (r-1)m\left(Q_{d-r}, r-1\right) + \sum_{j=1}^{\lceil r/2 \rceil - 1} \binom{r}{2j+1} m\left(Q_{d-r}, r-2j\right)$$

by construction. We will be done if we can show that A_0 percolates with respect to the r-neighbour bootstrap process.

We begin by showing that every vertex x with $[x]_r \in L_0 \cup L_1$ is eventually infected. First, we can infect every vertex x such that $[x]_r = (1, 0, \ldots, 0)$, one by one in some order, by definition of B_r. Next, consider a vertex x such that $[x]_r \in L_0$ and $[x]_{d-r} \in B_{r-1}$. Then x has $r-1$ neighbours $z \in A_0$ such that $[z]_r \neq (1, 0, \ldots, 0)$, by construction, and one infected neighbour y such that $[y]_r = (1, 0, \ldots, 0)$. Thus, every such x becomes infected. Now, by definition of B_{r-1}, the remaining vertices x such that $[x]_r \in L_0$ can be infected since every such vertex has an infected neighbour y such that $[y]_r = (1, 0, \ldots, 0)$. Finally, each vertex x such that $x \neq (1, 0, \ldots, 0)$ and $[x]_r \in L_1$ becomes infected using the definition of B_{r-1} and the fact that every vertex y with $[y]_r \in L_0$ is already infected.

Now, suppose that, for some $1 \leq j \leq \lceil r/2 \rceil - 1$ every vertex x such that $[x]_r \in L_0 \cup \cdots \cup L_{2j-1}$ is already infected. We show that every vertex x with $[x]_r \in L_{2j} \cup L_{2j+1}$ is eventually infected. First, consider a vertex x with $[x]_r \in L_{2j}$ and $[x]_{d-r} \in B_{r-2j}$. Such a vertex has $2j$ infected neighbours y such that $[y]_r \in L_{2j-1}$ and $r-2j$ neighbours z such that $[z]_r \in L_{2j+1} \cap A_0$. Therefore, every such x becomes infected. Now, by definition of B_{r-2j}, the remaining vertices x such that $[x]_r \in L_{2j}$ can be infected since every such vertex has $2j$ infected neighbours y such that $[y]_r \in L_{2j-1}$. Finally, each vertex x such that $[x]_r \in L_{2j+1}$ becomes infected using the definition of B_{r-2j} and the fact that every vertex y with $[y]_r \in L_{2j-1}$ is already infected.

Finally, if r is even, then we need to show that every vertex of L_r becomes infected. Every such vertex has precisely r neighbours in L_{r-1}. Thus, given that every vertex of L_{r-1} is infected, x becomes infected as well. This completes the proof. □

We remark that the recursion in Lemma 5.1 gives a bound of the form $m\left(Q_d, r\right) \leq \frac{(1+o(1))d^{r-1}}{r!}$ where the second order term is better than that of (1.2). Next, we prove Theorem 1.4.

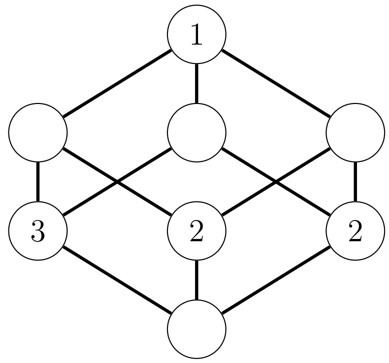

Figure 2: An illustration of the set A_0 constructed in the proof of Theorem 5.1 in the case $r = 3$. Each node represents a copy of Q_{d-3}. The set A_0 consists of a copy of B_i on each node labelled $i \in \{1, 2, 3\}$.

Proof of Theorem 1.4. The lower bound follows from Theorem 1.3. We prove the upper bound by induction on d. First, we settle the cases $d \in \{3, \ldots, 8\}$. For notational convenience, we associate each element v of $\{0,1\}^d$ with the of subset of $[d]$ for which v is the characteristic vector. Moreover, we identify each non-empty subset of $[d]$ with the concatenation of its elements (e.g. $\{1, 3, 7\}$ is written 137). One can verify (by hand or by computer) that the set A_0^d, defined below, percolates with respect to the 3-neighbour bootstrap process in Q_d and that it has the cardinality $\left\lceil \frac{d(d+3)}{6} \right\rceil + 1$.

$$
\begin{aligned}
A_0^3 &:= \{1, 2, 3, 123\},\\
A_0^4 &:= \left(A_0^3 \setminus \{3\}\right) \cup \{134, 4, 234\},\\
A_0^5 &:= \left(A_0^4 \setminus \{134\}\right) \cup \{135, 245, 12345\},\\
A_0^6 &:= \left(A_0^5 \setminus \{135, 245\}\right) \cup \{346, 12356, 456, 23456\},\\
A_0^7 &:= \left(A_0^6 \setminus \{346\}\right) \cup \{13457, 24567, 12367, 1234567\},\\
A_0^8 &:= \left(A_0^7 \setminus \{13457, 24567\}\right) \cup \{34568, 1234578, 34678, 25678, 2345678\}.
\end{aligned}
$$

Now, suppose $d \geq 9$ and that the theorem holds for smaller values of d. If d is odd, then we apply Lemma 5.1 to obtain

$$m\left(Q_d, 3\right) \leq m\left(Q_{d-3}, 3\right) + 2m\left(Q_{d-3}, 2\right) + m\left(Q_{d-3}, 1\right).$$

Clearly, $m\left(Q_{d-3}, 1\right) = 1$ and it is easy to show that $m\left(Q_{d-3}, 2\right) \leq \frac{d-3}{2} + 1$ (since $d - 3$ is even). Therefore, by the inductive hypothesis,

$$m\left(Q_d, 3\right) \leq \left\lceil \frac{(d-3)d}{6} \right\rceil + 1 + 2\left(\frac{d-3}{2} + 1\right) + 1 = \left\lceil \frac{d(d+3)}{6} \right\rceil + 1.$$

Now, suppose that $d \geq 10$ is even. For $t \in \{1, 2, 3\}$, let B_t be a subset of $V\left(Q_{d-6}\right)$ of cardinality $m\left(Q_{d-6}, t\right)$ which percolates with respect to the t-neighbour bootstrap process

on Q_{d-6} and let A_0^6 be as above. Given a vector $x \in V(Q_d)$, let $[x]_6$ be the restriction of x to its first six coordinates and $[x]_{d-6}$ be the restriction of x to its last $d-6$ coordinates. We define a subset A_0 of $V(Q_d)$. We include a vertex $x \in V(Q_d)$ in A_0 if $[x]_6 \in A_0^6$ and one of the following holds:

- $[x]_6 = (0,0,1,1,0,1)$ and $[x]_{d-6} \in B_3$.
- $[x]_6 \neq (0,0,1,1,0,1)$ and we have $x_5 = 1$ and $[x]_{d-6} \in B_2$.
- $x_5 = x_6 = 0$ and $[x]_{d-6} \in B_1$.

The fact that A_0 percolates follows from arguments similar to those given in the proof of Theorem 5.1; we omit the details. By construction,

$$|A_0| = m(Q_{d-6},3) + 4m(Q_{d-6},2) + 5m(Q_{d-6},1)$$

which equals

$$\left\lceil \frac{(d-6)(d-3)}{6} \right\rceil + 1 + 4\left(\frac{d-6}{2} + 1\right) + 5 = \left\lceil \frac{d(d+3)}{6} \right\rceil + 1$$

by the inductive hypothesis. The result follows. □

6 Concluding Remarks

In this paper, we have determined the main asymptotics of $m(Q_d, r)$ for fixed r and d tending to infinity and obtained a sharper result for $r = 3$. We wonder whether sharper asymptotics are possible for general r.

Question 6.1. *For fixed $r \geq 4$ and $d \to \infty$, does*

$$\frac{m(Q_d, r) - \frac{d^{r-1}}{r!}}{d^{r-2}}$$

converge? If so, what is the limit?

As Theorem 1.4 illustrates, it may be possible to obtain an exact expression for $m(Q_d, r)$ for some small fixed values of r. The first open case is the following.

Problem 6.2. *Determine $m(Q_d, 4)$ for all $d \geq 4$.*

Using a computer, we have determined that $m(Q_5, 4) = 14$, which is greater than the lower bound of 13 implied by Theorem 1.3. Thus, Theorem 1.3 is not tight for general d and r. However, we wonder whether it could be tight when r is fixed and d is sufficiently large.

Question 6.3. *For fixed $r \geq 4$, is it true that*

$$m\left(Q_d, r\right) = 2^{r-1} + \left\lceil \sum_{j=1}^{r-1} \binom{d-j-1}{r-j} \frac{j2^{j-1}}{r} \right\rceil$$

provided that d is sufficiently large?

Another direction that one could take is to determine $\operatorname{wsat}\left(G, S_{r+1}\right)$ for other graphs G. For example, one could consider the d-dimensional torus $\mathbb{Z}_n^d$.

Problem 6.4. *Determine* $\operatorname{wsat}\left(\mathbb{Z}_n^d, S_{r+1}\right)$ *for all n, d and r.*

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A Appendix: An Explicit Linear Algebraic Construction

Given integers k and ℓ with $k \geq \ell \geq 0$, we construct an explicit subspace X of $\mathbb{R}^k$ of dimension $k-\ell$ such that $|\text{supp}(x)| \geq \ell+1$ for every $x \in X \setminus \{0\}$. This can be seen as an alternative proof of Lemma 3.3.

The construction is based on a so called *Vandermonde matrix*. For $1 \leq i \leq k-\ell$ we let $\alpha_i \in \mathbb{R}^\ell$ be the vector such that, for $1 \leq j \leq \ell$,

$$\alpha_{i,j} := i^j.$$

Now, for $1 \leq i \leq k-\ell$ let e_i be the ith standard unit basis vector of $\mathbb{R}^{k-\ell}$ and define

$$v_i := \alpha_i \oplus e_i.$$

The space X is defined to be $\text{span}\{v_1, \ldots, v_{k-\ell}\}$. It is clear that $\dim(X) = k-\ell$ by construction. All that remains is to show that $|\text{supp}(x)| \geq \ell+1$ for every $x \in X \setminus \{0\}$. We require a few definitions.

Definition A.1. Given a set $T \subseteq [k]$, let $\pi_T : \mathbb{R}^k \to \mathbb{R}^{|T|}$ be the standard projection $\pi_T : (x_1, \ldots, x_k) \mapsto (x_i : i \in T)$.

Definition A.2. For $n \geq 1$, a collection $Z \subseteq \mathbb{R}^n$ is in *general position* in $\mathbb{R}^n$ if any set of at most n vectors from Z is linearly independent.

Our proof of the following proposition follows an argument of Moshonkin.[1]

Proposition A.3. *For any set $T \subseteq [\ell]$, the vectors $\{\pi_T(\alpha_i) : 1 \leq i \leq k-\ell\}$ are in general position in $\mathbb{R}^{|T|}$.*

Proof. We assume that $|T| \geq 1$; otherwise, the result is trivial. Let $t := |T|$. Suppose that the proposition is false and let $I \subseteq [k-\ell]$ be a set of cardinality t for which there exists $\{c_i : i \in I\}$, not all of which are zero, such that

$$\sum_{i \in I} c_i \pi_T(\alpha_i) = 0.$$

Equivalently, for each $j \in T$,

$$\sum_{i \in I} c_i i^j = 0.$$

Since the determinant of a square matrix is equal to the determinant of its transpose, there must also exist scalars $\{c'_j : j \in T\}$, not all zero, such that

$$\sum_{j \in T} c'_j i^j = 0$$

[1] A. G. Moshonkin, *Concerning Hall's theorem*, Mathematics in St. Petersburg, Amer. Math. Soc. Transl. Ser. 2, vol. 174, Amer. Math. Soc., Providence, RI, 1996, pp. 73–77.

for every $i \in I$.

Let $p(x)$ denote the real polynomial $\sum_{j \in T} c'_j x^j$. Then $p(x)$ is a polynomial with between one and t non-zero terms and at least t positive real roots (namely, each $i \in I$). We show, by induction on t, that no such polynomial can exist. The base case $t = 1$ is trivial. Now, let $p(x)$ be a polynomial with $t \geq 2$ non-zero terms and at least t positive real roots. Define $q(x)$ to be the polynomial of smallest degree such that $q(x) = x^s p(x)$ for some $s \geq 0$. It is clear that $q(x)$ has at least as many positive real roots as $p(x)$. However, the derivative of $q(x)$ has at most $t-1$ positive terms and at least $t-1$ positive real roots, contradicting the inductive hypothesis. This completes the proof. $\square$

Now, suppose that $x \in X \setminus \{0\}$ such that $|\mathrm{supp}(x)| \leq \ell$. Define

$$U_1 := \mathrm{supp}(x) \cap [\ell],$$

$$U_2 := \mathrm{supp}(x) \setminus [\ell] \text{ and}$$
$$T := [\ell] \setminus U_1.$$

Since $x \in X \setminus \{0\}$, we can write

$$x = \sum_{i=1}^{k-\ell} c_i v_i$$

for scalars c_i which are not all zero. However, it must be the case that $c_i = 0$ for each i such that $i + \ell \notin U_2$. Thus,

$$0 = \pi_T(x) = \sum_{i:i+\ell \in U_2} c_i \pi_T(v_i) = \sum_{i:i+\ell \in U_2} c_i \pi_T(\alpha_i). \tag{A.4}$$

However, since $|U_1| + |U_2| \leq \ell$ we have $|T| = \ell - |U_1| \geq |U_2|$. Thus, (A.4) contradicts Proposition A.3. It follows that $|\mathrm{supp}(x)| \geq \ell + 1$ for every $x \in X \setminus \{0\}$, as required.